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Beam Cavity Coupling Impedance

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I. Introduction

The purpose of this note is to formulate the beam-cavity coupling impedance in terms of the normal modes of the cavity. While the analysis will be carried out only for the longitudinal beam-cavity coupling impedance, it should be easily adapted to the transverse impedance. The hope is that such a formulation may illuminate the behavior of the coupling impedance in various frequency limits.

II. Review of Normal Mode Analysis[1)]

We shall obtain the normal modes for an ideal cavity with perfect walls, whose boundary is divided into the surface S, with the boundary condition

$$\vec{n} \times \vec{E}_\ell = \vec{n}\cdot\vec{H}_\ell = \vec{n} \times \vec{F}_m = 0 \qquad (S) \qquad (2.1)$$

and the surface S', with the boundary condition

$$\vec{n}\cdot\vec{E}_\ell = \vec{n} \times \vec{H}_\ell = \vec{n} \times \vec{F}_m = 0 \qquad (S') \qquad (2.2)$$

Here $\vec{n}$ is the (outward) normal at the boundary, $\vec{F}_m$ represents the irrotational modes needed for the expansion of fields in the presence of charge, and $\vec{E}_\ell$, $\vec{H}_\ell$ represent the solenoidal modes which are closest to the normal excitation modes of the cavity.

Specifically, we write the eigenvalue equation

1) This treatment follows that given in Slater, Microwave Analysis, Van Nostrand Press (1950)

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$$\left.\begin{aligned} \nabla^2\phi_m + k_m{}^2\phi_m &= 0 \\ \phi_m(S) = \phi_m(S') &= 0 \end{aligned}\right\} \tag{2.3}$$

$$\vec{F}_m = - \nabla\phi_m , \qquad \nabla \times \vec{F}_m = 0 \tag{2.4}$$

for the irrotational field and the eigenvalue equations

$$\begin{aligned} \nabla \times (\nabla \times \vec{E}_\ell) &= k_\ell{}^2\vec{E}_\ell \\ \nabla \times (\nabla \times \vec{H}_\ell) &= k_\ell{}^2\vec{H}_\ell \end{aligned} \tag{2.5}$$

$$\nabla \times \vec{E}_\ell = k_\ell\vec{H}_\ell , \qquad \nabla \times \vec{H}_\ell = k_\ell\vec{E}_\ell \tag{2.6}$$

$$\begin{aligned} \vec{n} \times \vec{E}_\ell(S) = \vec{n}\cdot\vec{H}_\ell(S) &= 0 \\ \vec{n} \times \vec{H}_\ell(S') = \vec{n}\cdot\vec{E}_\ell(S') &= 0 \end{aligned} \tag{2.7}$$

$$\nabla\cdot\vec{E}_\ell = \nabla\cdot\vec{H}_\ell = 0 \tag{2.8}$$

for the solenoidal fields. We shall accept Slater's assertion that $\vec{E}_\ell$, $\vec{F}_m$ form a complete set for the expansion of the electric field, and $\vec{H}_\ell$ forms a complete set for the expansion of the magnetic field. It is a simple matter to confirm the orthogonality relations (and to set the normalizations)

$$\int \vec{E}_\ell\cdot \vec{E}_{\ell'}\, dv = \int \vec{H}_\ell \cdot\vec{H}_{\ell'}\, dv = \delta_{\ell\ell'} \tag{2.9}$$

$$\int \vec{F}_m\cdot\vec{F}_{m'}\, dv = k_m^2 \int \phi_m\, \phi_{m'}\, dv = \delta_{mm'} \tag{2.10}$$

$$\int \vec{E}_\ell \cdot\vec{F}_m\, dv = 0 \qquad (k_\ell \neq k_m) \tag{2.11}$$

The general approach to using the normal modes in a particular application is to start with Maxwell's equations

$$\nabla\cdot\vec{H} = 0 \tag{2.12}$$
$$\nabla\cdot\vec{E} = \rho/\varepsilon \tag{2.13}$$

$$\nabla \times \vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \tag{2.14}$$

$$\nabla \times \vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t} + \vec{J} \tag{2.15}$$

where ε, μ are taken to be constant, and where all fields are assumed to depend on $\vec{x}$ and t.

Let us then expand $\vec{E}$ and $\vec{H}$ as

$$\vec{E}(\vec{x},t) = \sum_m U_m(t)\vec{F}_m(x) + \sum_\ell V_\ell(t)\vec{E}_\ell(\vec{x}) \tag{2.16}$$

$$\vec{H}(\vec{x},t) = \sum_\ell I_\ell(t)\vec{H}_\ell(\vec{x}) \tag{2.17}$$

where

$$\begin{aligned} U_m &= \int \vec{E}\cdot\vec{F}_m \, dv \\ V_\ell &= \int \vec{E}\cdot\vec{E}_\ell \, dv \\ I_\ell &= \int \vec{H}\cdot\vec{H}_\ell \, dv \end{aligned} \tag{2.18}$$

Equation (2.12) is satisfied automatically because of Eq. (2.8). If we multiply Eq. (2.13) by ϕ_m and integrate over the cavity volume, we find

$$- \int \phi_m \nabla\cdot \vec{E} \, dv = U_m = - \frac{1}{\varepsilon} \int \rho \, \phi_m \, dv \tag{2.19}$$

where we have used Eq. (2.3b) to eliminate the boundary terms in the integration by parts. Similarly we can (scalar) multiply Eq. (2.14) by $\vec{H}_\ell$, and Eq. (2.15) by $\vec{E}_\ell$ and integrate over the cavity volume to obtain

$$\int \vec{H}_\ell \cdot \nabla \times \vec{E} \, dv = \int_S dS \, \vec{n} \cdot \vec{E} \times \vec{H}_\ell + k_\ell V_\ell = -\mu \frac{\partial I_\ell}{\partial t} \tag{2.20}$$

$$\int \vec{E}_\ell \cdot \nabla \times \vec{H} \, dv = -\int_{S'} dS \, \vec{n} \cdot \vec{E}_\ell \times \vec{H} + k_\ell I_\ell = \varepsilon \frac{\partial V_\ell}{\partial t} + \int \vec{J} \cdot \vec{E}_\ell \, dv \tag{2.21}$$

In proceeding further we shall assume that all fields, charge and current have the time dependence $e^{-i\omega t}$, and obtain

$$k_\ell V_\ell - i\omega\mu \quad I_\ell = -\int_S dS \, \vec{n} \cdot \vec{E} \times \vec{H}_\ell \tag{2.22}$$

$$k_\ell I_\ell + i\omega\varepsilon \quad V_\ell = \int_{S'} dS \, \vec{n} \cdot \vec{E}_\ell \times \vec{H} + \int \vec{J} \cdot \vec{E}_\ell \, dv \tag{2.23}$$

$$U_m = \frac{1}{\varepsilon} \int \rho \phi_m \, dv = -\frac{i}{\omega\varepsilon} \int \vec{J} \cdot \vec{F}_m \, dv \tag{2.24}$$

where the last form of Eq. (2.24) is obtained using the equation of continuity for current and charge.

The surface integral in Eq. (2.22) is non-vanishing only because of wall losses. Assuming dominance of a particular mode, one can write approximately

$$\int_S dS \, \vec{n} \cdot \vec{E} \times \vec{H}_\ell \simeq \frac{\omega\mu\delta \quad (1-i)}{2} \int_S dS \, \vec{H} \cdot \vec{H}_\ell \tag{2.25}$$

where the skin depth δ is given by

$$\delta^2 = \frac{2}{\omega\mu\sigma} \tag{2.26}$$

and where we have assumed an outward penetration into the metal skin. If we define the quality factor of mode ℓ by

$$Q_\ell^{-1} \equiv \frac{\delta}{2} \int_S dS\, H_\ell^{\,2} \tag{2.27}$$

the surface integral can be written as

$$\int_S dS\, \vec{n}\cdot\vec{E}\times\vec{H}_\ell \simeq (1-i)\omega\mu \;\; I_\ell\, Q_\ell^{-1} \tag{2.28}$$

Thus Eq. (2.22) becomes

$$k_\ell\, V_\ell \simeq i\omega\mu \;\; I_\ell \left(1 + \frac{1+i}{Q_\ell}\right) \tag{2.29}$$

or

$$I_\ell \simeq \left(\frac{-ik_\ell}{\omega\mu}\right) V_\ell \left(1 - \frac{1+i}{Q_\ell}\right) \tag{2.30}$$

where we assume that $Q_\ell >> 1$. Using Eq. (2.30), we now find for Eq. (2.23)

$$\left\{ \omega^2 \mu\varepsilon - k_\ell^{\,2}\left(1 - \frac{i}{Q_\ell}\right) \right\} V_\ell =$$

$$= -i\omega\mu \int_{S'} dS\, \vec{n}\cdot\vec{E}_\ell \times\vec{H} - i\omega\mu \int \vec{J}\cdot\vec{E}_\ell \; dv \tag{2.31}$$

where we have absorbed the small real frequency shift due to wall losses into the definition of k_ℓ.

In using Eq. (2.31) we recognize it as a resonant high Q oscillator being driven by a current in the cavity $(\vec{J})$ as well as by any power source coupled through the surface S'. Note that in the absence of drive power, or current, the cavity resonates at the frequency

$$\omega \simeq k_\ell c\left(1 - \frac{i}{Q_\ell}\right)^{1/2} \simeq \omega_\ell - \frac{i\omega_\ell}{2Q_\ell} \tag{2.32}$$

Clearly the time dependence

$$e^{-i\omega t} \simeq e^{-i\omega_\ell t - (\omega_\ell t/2Q_\ell)} \tag{2.33}$$

implies a damped oscillation with the usual definition of Q_ℓ.

III. Coupling Impedance

A commonly employed definition of the coupling impedance[2)]

[2)] See, for example A.W. Chao, 1982 Summer School Lectures, SLAC for a ring of circumference $2\pi R$ is

$$Z_L(\omega) = -\frac{V_o}{I_o} \tag{3.1}$$

where the driving current is

$$J_z(\vec{x},t) = I_o \, e^{-i\omega t + i\omega z/v} \quad \delta(x-x_o)\;\delta(y-y_o) \tag{3.2}$$

and the resulting voltage is

$$V_o \, e^{-i\omega t} \equiv \int_0^{2\pi R} e^{-i\omega z/v} \; E_z(\vec{x},t)\big|_{x=x_o,y=y_o} \, dz \tag{3.3}$$

We here assume that the current element is at a fixed location x_o, y_o in the x, y plane. Equation (2.24) gives

$$U_m = -\frac{iI_o}{\omega\varepsilon} e^{-i\omega t} \int_0^{2\pi R} F_{mz} \, e^{i\omega z/v} \, dz \tag{3.4}$$

and Eq. (2.31) gives

$$V_\ell = -(iI_o\omega/\varepsilon)\, e^{-i\omega t} \frac{\int_0^{2\pi R} E_{\ell z}\, e^{i\omega z/v}\, dz}{(\omega^2 - \omega_\ell^2 + i\omega\omega_\ell\,/Q_\ell)} \tag{3.5}$$

where we have replaced $\omega_\ell{}^2/Q_\ell$ by $\omega_\ell\omega/Q_\ell$ to yield the usual circuit form for cavity impedance. Since Equation (3.3) corresponds to

$$V_o \, e^{-i\omega t} = \sum_m U_m \int_0^{2\pi R} F_{mz} \, e^{-i\omega z/v} dz + \sum_\ell V_\ell \int_0^{2\pi R} E_{\ell z} e^{-i\omega z/v} \, dz \tag{3.6}$$

we find for the longitudinal coupling impedance

$$Z_L(\omega) = (i\omega/\varepsilon) \sum_\ell \frac{|\int E_{\ell z} e^{-i\omega z/v} \, dz|^2}{\omega^2 - \omega_\ell^2 + i\omega\omega_\ell/Q_\ell} + (i/\omega\varepsilon) \sum_m |\int F_{mz} e^{-i\omega z/v} dz|^2 \tag{3.7}$$

A similar expression can be obtained for the transverse impedance.

IV. Special Limits

a) Resonance Behavior

If the coupling impedance is dominated by a single cavity resonance, one can write, in the vicinity of the resonance

$$\frac{Z_L(\omega)}{Z_0 Q_\ell} \simeq (c/\omega_\ell) \frac{|\int E_{\ell z}\, e^{-i\omega z/v}\, dz|^{\,2}}{1 + iQ_\ell(\omega_\ell/\omega \;\; - \omega\,/\omega_\ell)} \tag{4.1}$$

where $Z_0 = (\mu/\varepsilon)^{1/2}$ is the impedance of free space. This form for resonant coupling impedance agrees with the usual definition of shunt impedance as $(\text{voltage})^2/(\text{power})$ and of Q_ℓ as (stored energy) (frequency)/(power).

b) Beam Pipe

The modes in a beam pipe of cross sectional radius b and circumferential length $2\pi R$ are easily obtained neglecting the ring curvature. Assuming azimuthal symmetry, the TM modes have electric fields proportional to

$$\begin{aligned} E_{z\ell} &= E_0\, e^{ik_s z}\, J_0(p_t r/b) \\ E_{r\ell} &= -(ik_s b/p_t)\, E_0\, e^{ik_s z}\, J_1(p_t r/b) \end{aligned} \tag{4.2}$$

where s is an integer and

$$k_s = s/R \tag{4.3}$$

to ensure a single value for the fields after one trajectory around the circumference, and where p_t is the t^{th} zero of the Bessel function J_0. The eigenfrequency is given by

$$(\omega_\ell/c)^2 = (s/R)^2 + (p_t/b)^2 \tag{4.4}$$

The normalization condition in Eq. (2.9) requires

$$E_0^{\,2} = \frac{p_t^{\,2}\, c^2}{2\pi^2\, R\, b^4\, \omega_\ell^{\,2}\, J_1^{\,2}(p_t)} \tag{4.5}$$

A beam pipe of constant cross section is anamolous, in the sense that the eigenmodes $\vec{F}_m$ have the same eigenfrequencies as the eigenmodes $\vec{E}_\ell$. Specifically we find

$$F_{mz} = F_o\, e^{ik_s z}\, J_o(p_t r/b)$$
$$F_{mr} = (ip_t/k_s b)\, F_o\, e^{ik_s z}\, J_1(p_t r/b) \tag{4.6}$$

with the normalization

$$F_o^{\,2} = \frac{s^2\, c^2}{2\pi^2\, R^3\, b^2\, \omega_m^{\,2}\, J_1^{\,2}(p_t)} = E_o^{\,2}\, \frac{b^2 s^2}{R^2 p_t^{\,2}} \tag{4.7}$$

and with

$$(\omega_m/c)^2 = (s/R)^2 + (p_t/b)^2 \tag{4.8}$$

If we evaluate the longitudinal coupling impedance for the rotation harmonic n, we have

$$\omega = n\, \omega_o = nv/R \tag{4.9}$$

The form factor in Eq. (3.7) then becomes proportional to

$$\int_o^{2\pi R} e^{i(s-n)z/R}\, dz = 2\pi R\, \delta_{sn} \tag{4.10}$$

Thus, only the modes with s = n nodes in the longitudinal direction survive, and the sum over modes in Eq. (3.7) reduces to the single sum over t. Specifically,

$$\frac{\omega^2 - \omega_\ell^2}{c^2} = (n\beta/R)^2 - (n/R)^2 - (p_t/b)^2 = -\left(\frac{p_t^2}{b^2} + \frac{n^2}{\gamma^2 R^2}\right) \tag{4.11}$$

$$\frac{Z_L(n)}{nZ_o} = (4\pi^2 i/\beta) \sum_{t=1}^{\infty} RE_o^{\,2} \left\{\frac{b^2}{p_t^{\,2}} - \frac{\beta^2}{(p_t/b)^2 + (n/\gamma R)^2}\right\} \tag{4.12}$$

where the two terms in the bracket { } come respectively from the sum over m and ℓ. We have here neglected Q_ℓ^{-1} compared to 1 because of the non-resonant behavior of the cavity modes with respect to ω. Using Eq. (4.5) for $E_o^{\,2}$, one finds

$$\frac{Z_L(n)}{nZ_o} = (2i/\beta\gamma^2) \sum_{t=1}^{\infty} \frac{1}{\{p_t^{\,2} + (nb/\gamma R)^2\} J_1^{\,2}(p_t)} \tag{4.13}$$

Since the primary contributions to the sum over t come from large t, we can write

$$p_t \simeq \pi t, \quad p_t^{\,2} J_1^{\,2}(p_t) \simeq (2p_t/\pi) \simeq 2t \tag{4.14}$$

and obtain

$$\frac{Z_L(n)}{nZ_o} \simeq (i/\beta\gamma^2) \sum_{t=1}^{\infty} \frac{t}{t^2 + (nb/\pi\gamma\, R)^2} \tag{4.15}$$

which diverges, especially as $\gamma \to \infty$. Convergence can however be recovered if we choose a uniform beam of radius a and recognize that the fields E_z and F_z' have a radial dependence $J_o(p_t r/b)$. Thus Eq. (4.13) needs to be replaced by

$$\frac{Z_L(n)}{nZ_o} = (2i/\beta\gamma^2) \sum_{t=1}^{\infty} \frac{J_1^{\,2}(p_t a/b)}{p_t^{\,2}\, J_1^2(p_t)(p_t a/2b)^2} \tag{4.16}$$

where we have neglected the term in γ^{-2} inside the sum.

An explicit form for the sum in Eq. (4.16) can be obtained by expressing various functions of r as expansions in the complete set of functions $J_o(p_t r/b)$ in the region $0 \leq r \lesssim b$. Specifically, one can easily show that

$$-\ell n\, (x_>) = 2 \sum_{t=1}^{\infty} \frac{J_o(p_t x_1) J_o(p_t x_2)}{p_t^{\,2}\, J_1^2(p_t)} \tag{4.17}$$

where $x_>$ is the larger of x_1 or x_2. Integrating Eq. (4.16) over $x_1 dx_1 x_2 dx_2$ leads to

$$2 \cdot \sum_{t=1}^{\infty} \frac{J_1^2(p_t a/b)}{p_t^{\,2} J_1^2(p_t)(p_t a/2b)^2} = \ell n\, (b/a) + 1/4 \tag{4.18}$$

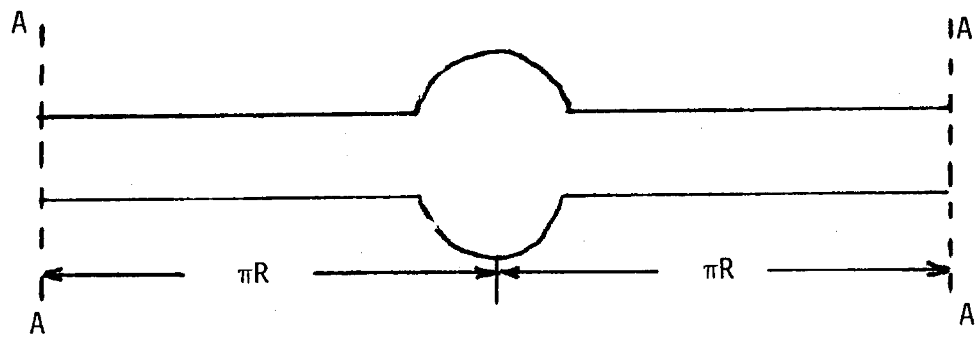
Thus the longitudinal coupling impedance of a wave guide of cross sectional radius b due to a beam of radius a is

$$\frac{Z_L(n)}{n\, Z_0} = (i/\beta\gamma^2)\ \{\ \ell n(b/a) + 1/4\ \} \tag{4.19}$$

in agreement with the well known result[3].

V. <u>Eigenmodes for a Cavity and Beam Pipe</u>

The evaluation of Eq. (3.7) for the coupling impedance for arbitrary ω appears to require knowledge of the entire eigenmode spectrum for the cavity. Since this will in general not be practical, we will discuss some simple cases for a single cavity connected to a beam pipe, as shown in the figure[4].



A) Below the cutoff of the beam pipe for TM modes the only eignemodes of the cavity-beam pipe system are essentially those of the isolated cavity. The coupling to the beam pipe will lead to a frequency shift proportional to the cube of the beam pipe radius, and to a change in the voltage integral which is proportional to the first power of the beam pipe radius.

B) Above the cutoff of the lowest TM mode in the beam pipe the field will exist primarily in the beam pipe, with the cavity acting to reflect

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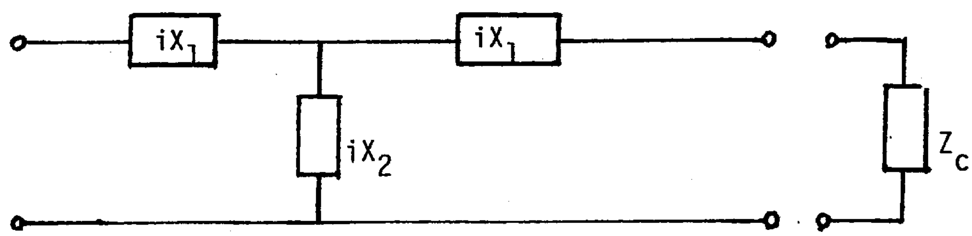
3) Nielsen, Sessler and Symon, Proc. of the Int'l. Conf. on High Energy Accelerators, Geneva, 1959, p. 239.

4) The case where the cavity is cylindrical, with radius larger than the beam pipe has been treated by Keil and Zotter, Particle Accelerators <u>3</u>, 11 (1972) and by Warnock, Bart and Fenster, Particle Accelerators <u>12</u>, 179 (1982)

and transmit the propagating mode. The eigenfrequencies of the cavity-beam pipe system are those for which the reflected and transmitted waves coincide with the incident waves after one traversal of the circumference. For a long circumference there will be many closely spaced eigenmodes whose frequencies will be determined by the reflection and transmission coefficients which can be expected to vary slowly with frequency.

The most general equivalent circuit for a symmetric obstacle (cavity) in a guide with a single propagating mode is shown in the figure:

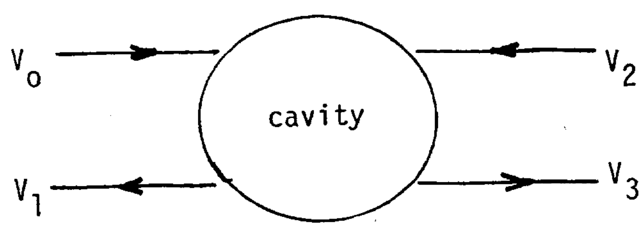


where Z_c is the characteristic impedance of the propagating mode in the guide. Simple circuit analysis leads to the reflection and transmission coefficients

$$R = -(1/2)\,\frac{Z_c - i(X_1+2X_2)}{Z_c + i(X_1+2X_2)} - (1/2)\frac{Z_c - iX_1}{Z_c + iX_1} \qquad (5.1)$$

$$T = -(1/2)\{\quad\downarrow\quad\} + (1/2)\{\quad\downarrow\quad\} \qquad (5.2)$$

If we consider the propagating waves as shown in the figure



we can write

$$V_1 = R\,V_o + T\,V_2 \qquad (5.3)$$
$$V_3 = R\,V_2 + T\,V_o$$

Continuity of the fields after one traversal of the circumference requires

$$V_2 = V_1\, e^{ik_\ell 2\pi R} \qquad (5.4)$$
$$V_o = V_3\, e^{ik_\ell 2\pi R}$$

where

$$k_\ell^{\,2} = (\omega_\ell/c)^2 - (p/b)^2 \qquad (5.5)$$

and where p = 2.405 is the first zero of $J_o(x)$. Combining Eqs. (5.3) and (5.4) leads to the eigenfrequency conditions

$$e^{-2\pi i k_\ell R} = \begin{cases} T + R, & \text{symmetric modes} \\ T - R, & \text{antisymmetric modes} \end{cases} \qquad (5.6)$$

Using Eqs. (5.1) and (5.2), we can write

$$k_\ell R = \ell + \tfrac{1}{2} + 2\tan^{-1}\left(\frac{X_1 + 2X_2}{Z_c}\right), \text{ symmetric modes} \qquad (5,7)$$

$$k_\ell R = \ell - 2\tan^{-1}(X_1/Z_c), \qquad \text{antisymmetric modes} \qquad (5,8)$$

where ℓ runs over all integers. Equations (5.7), (5.8), and (5.1) therefore determine the eigenfrequencies of the cavity-beam pipe system.

In the above discussion, we have assumed that cavity resonances are well spaced and do not coincide with those obtained from Eqs. (5.7) and (5.8). If this is not the case, x_1 and x_2 will exhibit singular behavior corresponding to fields being much larger in the cavity than in the beam pipe.

A similar analysis must be performed for the $\vec{F}_m$ type modes of Eqs. (2.3) and (2.4). In addition it is necessary to have expressions for the fields in order to evaluate the integrals in Eq. (3.7). Only then can one try to evaluate the coupling impedance, and particularly the changes brought about by the presence of the cavity.

C) Let us suppose that we have solved the electromagnetic field problem for a single symmetric propagating mode, whose frequency is given by Eq. (5.7). The electric field components can be written, following Eq. (4.2), as

$$\begin{aligned} E_{z\ell} &= E_o \,\{\cos(k_\ell\cdot(z-\pi R)\,)\; J_o(pr/b) + \delta E_z\,\} \\ E_{r\ell} &= E_o \,\{(k_\ell b/p)\,\sin(k_\ell\cdot(z-\pi R)\,)\; J_1(pr/b) + \delta E_r\} \end{aligned} \tag{5.9}$$

where $E_o\delta E_z$, $E_o\delta E_r$ are the components of the difference between the fields of the propagating mode and the actual fields. Clearly δE_z, δE_r are most prominent in the cavity and decay exponentially as one enters the beam pipe.

The normalization is determined by $\int E_\ell{}^2 dv = 1$. This leads to

$$\begin{aligned} E_o{}^2 \,\{ &(\pi b^4/p^2) J_1^2(p) < \pi R(\frac{p^2}{b^2} + k_\ell{}^2) + (\frac{p^2}{b^2} - k_\ell{}^2)\frac{\sin(2k_\ell\pi R)}{2k_\ell} > + \\ &+ 8\pi\cos(k_\ell\pi R) \iint_{\text{half cavity}} rdrdz < J_o(pr/b)\cos(k_\ell z)\delta E_z - \frac{k_\ell b}{p} J_1(\frac{pr}{b})\sin(k_\ell z)\delta E_r > \\ &+ 4\pi \iint_{\text{half cavity}} rdrdz \;<(\delta E_z)^2 + (\delta E_r)^2> \;\} = 1 \end{aligned} \tag{5.10}$$

where the integrals extend over the half cavity and that portion of the beam pipe in which δE_z and δE_r are significant. For R>>b, Eq. (5.10) is of the form

$$E_o^{\ 2}\ (\ \alpha R + \beta\ ) = 1 \tag{5.11}$$

Similarly, the voltage integral averaged over the beam

$$\left|\int_o^{2\pi R} dz\ E_{\ell z}\ e^{-i\omega z/v}\right| = 2E_o\ \left|(2/a^2)\int_o^a rdr\int_o^{\pi R} du\ \{\cos(\frac{\omega u}{v})\cos(k_\ell u)J_o(\frac{pr}{b})\}+\right.$$

$$\left. + (2/a^2)\ \int_o^a rdr\ \int_o^\infty dz\ \ \delta E_z(z,r)\ \cos\frac{\omega(z-R)}{v}\right| \tag{5.12}$$

can be written, for $\omega = nv/R$, as

$$\left|\int_o^{2\pi R} dz E_{\ell z} e^{-inz/R}\right| = 2E_o\ \left|R\ \frac{J_1(\frac{pa}{b})k_\ell R\sin(k_\ell \pi R)}{(\frac{pa}{2b})\{(k_\ell R)^2-n^2\}} + \int_o^\infty dz\overline{\delta E}_z\cos\frac{nz}{R}\ \right| \tag{5.13}$$

where $\overline{\delta E}_z$ is the average of δE_z over the uniform beam of radius a. For R>>b, Eq. (5.13) is of the form

$$\left|\int_o^{2\pi R} dz\ E_{\ell z}\ e^{-i\omega z/v}\ \right| = E_o(\gamma R + \varepsilon) \tag{5.14}$$

Similar expressions will be valid for the $\vec{F}_m$ type fields as well.

One can now use Eqs. (5.11) and (5.14) to see the structure of the coupling impedance. Specifically, one finds

$$E_o^2(\gamma R + \varepsilon)^2 = \frac{(\gamma R+\varepsilon)^2}{\alpha R+\beta} \simeq (\gamma^2/\alpha)R + (\frac{2\gamma\varepsilon}{\alpha} - \frac{\beta\gamma^2}{\alpha^2}) \tag{5.15}$$

enabling us to conclude that the correction to $Z_L(n)/nZ_o$ in Eq. (4.19) will be of order R^{-1}. This is in agreement with the result of Keil and Zotter[4)] for a short cylindrical cavity.

It should be pointed out that we have evaluated Eq. (3.7) for the coupling impedance only for the eigenmodes for which a single mode can propagate in the beam pipe. In this case we have expressed the eigenfrequencies in Eqs. (5.5), (5.7), (5.8) in terms of two impedance parameters. It should also be possible to express δE_z and δE_r in terms of the modes of the isolated cavity. However the result in Eq. (4.19) suggests that we will need to sum over many modes, including ones in which more than one mode can propagate in the beam pipe. Although we have performed this sum for the beam pipe alone, it is not clear whether the higher modes will be important in evaluating the effect due to the presence of the cavity.

VI. Summary and Conclusions

We have formally identified the eigenmodes of the cavity-beam pipe configuration and have obtained a general expression (Eq. (3.7)) for the longitudinal coupling impedance. This reduces to the usual resonant result for an isolated cavity. The infinite sum over modes can also be evaluated for a beam pipe alone, giving the usual result[3].

The eigenmodes of the cavity-beam pipe system have been explored when only one mode can propagate in the beam pipe. The structure of the result suggests that it may be possible to obtain the contribution of the cavity to the coupling impedance by evaluating a limited number of eigenmodes of the cavity-beam pipe system. Analytic techniques using variational principles can be developed for this evaluation, but the use of computer programs like SUPERFISH may be simpler.

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